Establishment of Statistical Damage Constitutive Model for Rock

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ABSTRACT

Based on Mohr theory, and adopt parabola as Mohr envelope, the condition that rock failure to meet is deduced and the expression method of rock micro-elements strength failure is established. A damage constitutive model is established by using statistical damage theory and rock triaxial stress-strain curve, and the model can reflect the full process of rock failure. Moreover, the impact of the parameters $n$ and $\tau_0$ in Weibull distribution on rock constitutive model is analyzed. In order to obtain a more practical damage constitutive model, based on the relationship between parameters and confining pressure, the parameters $n$ and $\tau_0$ are modified by linear fitting. A comparative analysis is performed between the corrected model, uncorrected model and the trial model. The results show that the corrected model is closer to the experimental results; this embodies the rationality of the model.

KEYWORDS: statistical damage theory; rock; linear fitting; Weibull distribution; micro-elements strength; stress-strain curve

INTRODUCTION

Rock is a kind of commonly used engineering materials; the constitutive model of rock is always the key point of the rock mechanics professors' attention. Since the rock complete stress-strain curve is put forward, a more suitable rock constitutive model under loading damage is studied by scholars at home and abroad. The stress and strain relationship in rock failure process can be correctly reflected by an appropriate constitutive relation. For a long time, the damage statistical mechanics is applied to study on rock constitutive relation [1-9]. At present, there are mainly two ways to research on damage constitutive relation [10]: One is based on the experiment, assuming that rock damage variable and stress-strain under loading obeys a certain relationship, thus a damage constitutive relation is established [11-12]; Another is based on the characteristics of the random distribution of rock micro-elements strength, a relationship between damage variable and stress-strain is established, then rock damage constitutive model is established[4,13-14]. Currently, the second method has a widest application. From the
distribution laws of micro-elements strength obedience, the Weibull distribution, normal distribution, lognormal distribution and power function distribution is used most commonly. In this paper, based on a certain distribution that the internal micro-elements failure obeys, a corresponding model is established for analyzing the rock failure process. In paper [4], with strain as the condition of micro-elements damage, a complete stress-strain constitutive relationship of the rock damage statistics is obtained; however, it can not well reflect the characteristics of micro-elements strength. By using statistical damage theory with the Drucker - Prager strength criterion as the micro-elements damage condition, statistical damage softening constitutive model is established in paper [1,5,10] to simulate the rock deformation and failure under different confining pressure; but the application of Drucker - Prager strength criterion is limited for its formula is more complex and conservative. In paper [15], a rock damage statistical constitutive model is obtained with the condition that rock micro-elements intensity achieves the maximum shear stress, but the softening characteristics after the peak of damage under the loading cannot be reflected well.

Based on research work made by pioneers, with the condition that the rock damage when shear stress reaches the maximum value, and the Mohr envelope is parabola, a new strength condition of rock failure is deduced and a expressive methods of rock micro-element strength failure is established. Base on the relationship between the model and confining pressure, a rock damage softening constitutive model is established, its can reflect the rock deformation and failure under the different confining pressure from the angle of the rock micro-elements strength obeys Weibull distribution. And the model is been corrected. A comparative analysis is performed between the corrected model, uncorrected model and the trial model. The results show that the corrected curve is closer to the trial curve; this embodies the rationality of the model.

ESTABLISHMENT OF ROCK DAMAGE STATISTICAL CONSTITUTIVE MODEL

Establishment of damage evolution equation

Natural rock is a kind of heterogeneous material in nature. The inside of rock contains a lot random defects, like pores, weak planes, holes and so on. And the micro-defects are a typical performance of damage. Define damage variable D as the ratio of the destroyed micro-elements number n under the exterior loading and the total micro-elements number N, and assume that the P[F] is the distribution density of the failure probability of rock micro-elements with the change of the strength F, then the failure probability of the rock damage variable is as follows[3]:

\[ D = \frac{n}{N} = \int_{0}^{F} P(x)dx \]  

(1)

The range of D is from 0 to 1.
Rock material exist a number of different strength weak links, and the damage under loading is a continuous process. Before destroying rock micro-elements obey Hooke's law, and rock material is isotropic on the macro. Now the strength of micro-elements is assumed to obey Weibull distribution [1, 4]. According to the former assumption on the internal strength probability distribution, when the rock shear stress is \( \tau_{\max} \), rock will be destroyed, then the failure probability density function of micro-elements is as follows:

\[
P(\tau) = \frac{n}{\tau_0} \left( \frac{\tau}{\tau_0} \right)^{n-1} \exp\left[ -\left( \frac{\tau}{\tau_0} \right)^n \right]
\]

In the formula above, \( \tau \), the maximum shear stress rock can withstand, represents failure strengths. Micro-elements will be damaged when failure strengths obtain the maximum shear. \( n \) and \( \tau_0 \) are parameters in Weibull distribution, they reflect the material damage mechanical property. The damage variable can be calculated by substituting the formula (2) into formula (1).

\[
D = \int_0^F P(x) dx = 1 - \exp\left[ -\left( \frac{\tau}{\tau_0} \right)^n \right]
\]

The formula reflects the process that material micro-elements is damaging until macroscopic is damage, this embodies the essence of material damage is a gradually accumulate process of internal micro-elements structure damage. The key to calculate the damage variable is to obtain a equation of \( \tau \). The solving process of micro-elements strength is described below.

**Determination of micro-elements strength**

The determination of micro-elements strength is mainly based on Mohr-Coulomb criterion [3], Drucker-Prager strength criterion [1, 10], etc. In fact, the shear stress increases with the increase of confining pressure when the rock is broken. The rock failure is mainly shear failure. The Mohr strength envelope can be fitted by a variety of curve equation. In this paper, a simple and widely applied parabolic curve is used to fit. The Mohr's stress circle is shown in figure 1, and the parabolic curve expression is as formula (4):

\[
\tau^2 = m \sigma_t (\sigma + \sigma_t)
\]

\( \sigma_t \) is the tensile strength of the rock, \( m \) is the opening size of the Parabolic Mohr envelope.
According to the processing method in article [15], when the principal stress is \( \sigma_1 \), the maximum shear stress is as follows,

\[
\tau_{\text{max}} = \left( \sigma_1 - \sigma_3 \right) / 2 \quad (5)
\]

The Mohr’s stress circle is determined by \( \sigma_1 \) and \( \sigma_3 \) (\( \sigma_{\text{max}} = \sigma_1, \sigma_{\text{min}} = \sigma_3 \)). The circle is tangent with the Mohr parabola envelope (formula (4)). And the tangent point of Mohr stress circle and the stress value of the stress circle center can be calculated. The expression of the stress value is formula (6):

\[
C = \left( \sigma_1 + \sigma_3 \right) / 2 \quad (6)
\]

From the parabola line (formula (4)) knows that the slope of the tangent point \( (\sigma, \tau) \) is as formula (7):

\[
\frac{d\tau}{d\sigma} = \frac{m\sigma_i}{2\tau} \quad (7)
\]

The slope of the line from the tangent point \( (\sigma, \tau) \) to the stress circle center \( (C,0) \) is as formula (8):

\[
k = \frac{\tau}{\sigma - C} = \frac{2\tau}{2\sigma - \sigma_1 - \sigma_3} \quad (8)
\]

So here is

\[
\frac{d\tau}{d\sigma} = -\frac{1}{k} \quad (9)
\]

Based on formula (7), (8) and (9):

\[
\sigma = \frac{\sigma_1 + \sigma_3 - m\sigma_i}{2} \quad (10)
\]
Substitute the formula (10) into formula (4):

$$\tau = \sqrt{\frac{1}{2} m \sigma_i (\sigma_i + \sigma_3 - m \sigma_i + 2 \sigma_i)}$$  \hspace{1cm} (11)$$

The formula (11) is the condition that the shear failure to meet. The damage evolution equation is calculated by substituting the formula (11) into formula (3):

$$D = 1 - \exp \left[ - \left( \frac{\sqrt{\frac{1}{2} m \sigma_i (\sigma_i + \sigma_3 - m \sigma_i + 2 \sigma_i)}}{\tau_0} \right)^n \right]$$ \hspace{1cm} (12)$$

From the formula (12) knows that the damage variable is related to the tensile strength, it also has a close relationship with confining pressure.

**Establishment of rock damage constitutive model**

Based on the Hypothesis of Strain Equivalence from J. Lemaitre[16], the constitutive relationship of rock damage is as formula [2-5] (13):

$$\sigma' = \sigma / (1 - D) = E \epsilon / (1 - D)$$ \hspace{1cm} (13)$$

$$\sigma'$$ is the effective stress, E is elastic matrix of rock materials, \(\epsilon\) is strain, \(\sigma\) is nominal stress of rock materials, and \(D\) is the damage variable.

Under the three-dimensional stress state,

$$\sigma_i = E \epsilon_i (1 - D) + \mu (\sigma_2 + \sigma_3)$$ \hspace{1cm} (14)$$

When \(D = 0\), the formula (14) is the equal of Hooke's law under triaxial stress state.

The micro-elements strength of rock damage is expressed in effective stress, in the conventional triaxial test the expression of the micro-elements strength is as formula (15):

$$\tau = \sqrt{\frac{1}{2} m \sigma_i' (\sigma_i' + \sigma_3' - m \sigma_i' + 2 \sigma_i')}$$ \hspace{1cm} (15)$$

Nominal stress \(\sigma_1, \sigma_2, \sigma_3\) and strain \(\epsilon_i\) can be measured in conventional triaxial test, among them, the surrounding rock \(\sigma_2 = \sigma_3\) the corresponding effective stress is \(\sigma_1', \sigma_2'\) and \(\sigma_3'\) respectively, as well \(\sigma_2' = \sigma_3'\). Based on the formula (14),

$$1 - D = \frac{1}{E \epsilon_i} (\sigma_1 - 2 \mu \sigma_3)$$ \hspace{1cm} (16)$$
\[\sigma'_i = \sigma_i/(1-D) \quad (17)\]

The micro-elements strength expressed in effective stress can be obtained by substituting the formula (17) into formula (15).

\[\tau = \frac{E\varepsilon_i}{\sigma_i - 2\mu\sigma_3} \left[ \frac{1}{2} m\sigma_i (\sigma_i + \sigma_3 - m\sigma_3 + 2\sigma_i) \right] \quad (18)\]

The damage evolution equation is calculated by substituting the formula (18) into formula (3):

\[D = \int_0^x P(x)dx = 1 - \exp \left[ - \left( \frac{E\varepsilon_i}{\sigma_i - 2\mu\sigma_3} \left[ \frac{1}{2} m\sigma_i (\sigma_i + \sigma_3 - m\sigma_3 + 2\sigma_i) \right] / \tau_0 \right)^n \right] \quad (19)\]

The triaxial stress-strain damage constitutive equation can be converted into formula (20),

\[\sigma_i = E\varepsilon_i (1-D) + \mu(\sigma_2 + \sigma_3) \quad (20)\]

The three-dimensional damage constitutive model is expressed by substituting the formula (19) into formula (20):

\[\sigma_i = E\varepsilon_i \exp \left[ - \left( \frac{E\varepsilon_i}{\sigma_i - 2\mu\sigma_3} \left[ \frac{1}{2} m\sigma_i (\sigma_i + \sigma_3 - m\sigma_3 + 2\sigma_i) \right] / \tau_0 \right)^n \right] + \mu(\sigma_2 + \sigma_3) \quad (21)\]

**SOLUTION AND ANALYSIS OF CONSTITUTIVE MODEL PARAMETERS**

**Solution for parameters \( n \) and \( \tau_0 \)**

From the statistical damage constitutive relationship (the formula (21)) can be known that only \( n \) and \( \tau_0 \) is unknown. The value of \( n \) and \( \tau_0 \) can be calculated by line fitting according to reference [1].
\[
\frac{\sigma_i - 2\mu \sigma_3}{E \varepsilon_i} = \exp\left[-\left(\frac{\tau}{\tau_0}\right)^n\right] \tag{22}
\]

Take the logarithm on both sides:

\[
\ln\left(\frac{\sigma_i - 2\mu \sigma_3}{E \varepsilon_i}\right) = -(\tau/\tau_0)^n \tag{23}
\]

Take the logarithm on both sides again:

\[
\ln\left[-\ln\left(\frac{\sigma_i - 2\mu \sigma_3}{E \varepsilon_i}\right)\right] = n \ln \tau - n \ln \tau_0 \tag{24}
\]

In formula (24):

\[
x = \ln \tau
\]

\[
y = \ln\left[-\ln\left(\frac{\sigma_i - 2\mu \sigma_3}{E \varepsilon_i}\right)\right]
\]

\[
a = -n \ln \tau_0 \tag{25}
\]

Then

\[
y = nx + a \tag{26}
\]

The parameters \( n \) and \( a \) are obtained by linear regression, then the value of \( \tau_0 \) is calculated by \( \tau_0 = \exp\left(-\frac{a}{n}\right) \), lastly, the constitutive relationship of complete stress-strain is obtained by substituting the parameters \( n \) and \( a \) into formula (21).

The value of the parameters \( n \) and \( \tau_0 \) corresponding to the trial curve in reference [14] can be calculated by line fitting (table 1). The three-dimensional damage constitutive equation that simulated the rock failure can be obtained by substituting the parameters into the relevant formula. The Simulation curve is shown in figure 2. From table 1, with the increase of confining pressure the value of \( n \) decreases, but the \( \tau_0 \) increases.

<table>
<thead>
<tr>
<th>confining pressure ( \sigma_3 ) /MPa</th>
<th>( n )</th>
<th>( \tau_0 )/MPa</th>
<th>correlation coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.9654</td>
<td>101.8437</td>
<td>0.9732</td>
</tr>
<tr>
<td>5</td>
<td>3.5163</td>
<td>136.8721</td>
<td>0.9815</td>
</tr>
<tr>
<td>10</td>
<td>3.0342</td>
<td>175.4329</td>
<td>0.9849</td>
</tr>
<tr>
<td>20</td>
<td>2.4889</td>
<td>228.4278</td>
<td>0.9753</td>
</tr>
</tbody>
</table>
As can be seen from Figure 2, firstly, with the increase of the confining pressure, the peak value of rock stress-strain curve increases, and the strain increases when the rock fails; secondly, after the complete stress-strain curves achieve the peak, different curves with different forms; thirdly, before the peak the linear part of the complete stress-strain curves basically does not change with the change of the confining pressure. These characteristics are consistent with the conclusions of a large number of trial curves. These show that the model can well simulate the complete stress-strain.
Impact of parameters $n$ and $\tau_0$ on damage statistical constitutive model

From formula (21), the parameters $n$ and $\tau_0$ are main factors affecting the constitutive relation, so here mainly introduces the impact of the parameters $n$ and $\tau_0$ on the constitutive model. By using the single factor analysis method, when $n$ remains unchanged, the complete stress-strain curves are show as figure 3 of the $\tau_0$ is 126MPa, 136MPa, 146MPa, 156MPa and 166MPa, respectively; When $\tau_0$ remains unchanged, the complete stress-strain curves are show as figure 4 of the $n$ is 2, 2.5, 3, 3.5 and 4, respectively.

From figure 3 and figure 4, the parameters $n$ and $\tau_0$ in the statistical damage constitutive model reflected the mechanical characteristics of the rock damage [17]. The peak of the complete stress-strain curves increases with the increases of the $n$ and $\tau_0$, however, the change of $n$ and $\tau_0$ has no impact on the linear part of the complete stress-strain curves before the peak, but has a obvious impact on the curves after the peak, and also can change the forms of the curves. Along with the increase of $\tau_0$, the strain of the rock failure increases obviously, but with the increase of $n$, the strain of the rock failure decreases. At the same time, with the increase of $n$, the more steep stress-strain curves are, the greater the rock brittleness.

![Figure 3](image)

**Figure 3**: Relation curve of rock damage model along with parameter $\tau_0$.
CORRECTION OF ROCK DAMAGE STATISTICAL CONSTITUTIVE MODEL

Based on the analysis in front, the model can fully reflect the complete stress-strain process of rock failure, especially the softening characteristics after the peak. However, the establishment of the model is only based on one trial curve under different confining pressure, so the model cannot reflect the comprehensive conditions of the multiple trial curves [10]. Thus, this constitutive model should be corrected appropriately.

The correcting must be able to reflect the characteristics of the rock failure process. From the previous analysis can be seen the parameters $n$ and $\tau_0$ have a great impact on the model, and the parameters $n$ and $\tau_0$ are calculated under different confining pressure situation. If the relationship between $n$, $\tau_0$ and confining pressure can be set up, there will be a reasonable way to correct the rock statistical damage constitutive model. The parameters $n$ and $\tau_0$ under different confining pressure are corrected according to the literature [5]. The scatter distribution of $n-\sigma_3$ and $\tau_0-\sigma_3$ is obtained by confining pressure as X-axis and $n$ and $\tau_0$ as Y-axis. Then the linear fitting is used and the fitting curves are shown in figure 5 and 6.
Figure 5: Relation curve of parameter $n$ along with confining pressure $\sigma_3$.

From Figure 5 and 6:

$$n = -0.0367\sigma_3 + 3.8948$$

$$\tau_0 = 3.1307\sigma_3 + 106.8560$$  \hspace{1cm} (27)$$

The correlation coefficient is 0.9689 and 0.9859, respectively, and the fitting effect is good. The corrected three-dimensional damage constitutive model of the complete stress-strain curves under different confining pressure can be obtained by substituting the formula (27) into formula (21).
From formula (28) knows that the different confining pressure can be used to fit the rock complete stress-strain curve.

**CALCULATION AND ANALYSIS**

Based on the material parameter data in literature [14], the rock Elastic modulus $E=90.0\text{GPa}$, Poisson ratio $\mu = 0.15$, internal friction angle $\phi = 30.3^\circ$, and take $m=5$. A comparative analysis is performed between the corrected model, uncorrected model and the trial model to verify the rationality of the correcting constitutive model. The trial curves that the confining pressure is 0MPa, 5MPa, 10MPa and 20MPa are fitted and then analyzed by the uncorrected formula (21) and the corrected formula (28). The comparison results are shown in figure 7.

From Figure 7, the uncorrected curve and the corrected curve have the following advantages than the trial curve. firstly, the softening characteristics can be reflected good; secondly, the rock
strength increases with the increases of confining pressure; lastly, the constitutive model not only can simulate the linear part of the curve before the peak, also can simulate the curve characteristic good after the peak, further more the complete stress-strain curve can be reflected reasonably. But the uncorrected constitutive relation is a result just under a specific confining pressure, it do not have general. The corrected curve can better reflect the constitutive relation of the rock statistical damage. At the same time, the corrected curve is closer to the trial curves; this embodies the rationality of the model.

CONCLUSION

The condition that rock failure to meet is deduced based on Mohr theory and Mohr envelope of the stress circle is parabola. From the angle of the rock micro-elements strength obeys Weibull distribution, the expression method of rock micro-elements strength failure is established, and the damage evolution equation of rock failure and the constitutive relation of load damage are obtained.

The impact of parameters $n$ and $\tau_0$ in Weibull distribution on the rock damage constitutive model is analyzed by using the single factor analysis method. The researches show that the peak of the complete stress-strain curves increases with the increases of $n$ and $\tau_0$, however, the change of $n$ and $\tau_0$ has no impact on the linear part of the complete stress-strain curves before the peak, but has a obvious impact on the curves after the peak; and with the increase of $n$, the more steep stress-strain curves are, the greater the rock brittleness.

Based on the relationship between the confining pressure and the parameters, the parameters $n$ and $\tau_0$ are corrected reasonably by using the linear fitting. The constitutive model of rock failure is more consistent to practical. The corrected model can fully reflect the characteristics of the complete stress-strain curve with the change of the stress state and the change of Weibull parameters $n$ and $\tau_0$.

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