Modified Model MGM (1, n) with Non-Equidistance and Multivariable Based on the Background Value Constructed by Gradual Optimization

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ABSTRACT
As a scientific theory on the poor information, grey system theory has wide adaptation. The background value of modified grey model MGM (1, n) with non-equidistance and multivariable should be constructed by gradually optimizing modelling method. Moreover, the above model MGM (1, n) should be established with the minimum mean of relative error as the target function, and the corresponding initial correction as design variable. Applied in modelling of both equal distance and non-equal distance, this model expands the application scope of grey model for its high precision and simple use. Moreover, examples show the practicability and reliability of the model.

KEYWORDS: Multiple variables, background value, non-equidistant sequence, gradual optimizing modelling, MGM (1, n) model, least square method

INTRODUCTION
Grey system theory is a theory solving the analysis, modeling, prediction, decision, and control of grey system. As the important part of grey system theory, grey model can search for the law of data by processing, rather than statistical law and probability distribution. This model not only supplements the deficiencies of data mining method, but also provides new scientific methods. Since the first proposal of grey system theory by Professor Deng Julong, the grey model is widely applied in various fields. There are various types of grey models, mainly including GM (1, 1), GM (1, N), MGM (1, N), etc. [1-3] GM (1, 1) is applied widely and studied deeply. The construction method of background value is the key factor affecting prediction accuracy and adaptation, while the optimization of its background value is one way for model modification. In order to improve the fitting and prediction accuracy of GM (1, 1) model, References [4-7] proposed various construction methods of background value, and also established various non-equidistant GM (1, 1) models. However, in social, economic and engineering systems, there are always various variables with internal connection. Therefore, GM (1, 1) can only be used for qualitative analysis, rather than as a prediction model [1]. MGM (1, N) model is the generalization of GM (1, 1) model with n-meta variable, but it is not a simple combination of GM (1, 1) models. Different from GM (1, n) model which only establishes a single n-meta first-order differential equation, MGM (1, N) model can make
solution by simultaneously establishing n n-meta differential equations. In this way, the model can reflect the mutual effects and interaction among various variables with parameters [8]. However, MGM (1, n) which was studied much less than GM (1, N) requires further research. Reference [2] established optimizing MGM (1, N) model by correcting the first component in sequence $x^{(1)}$, which is the initial condition of grey differential equation. According to the new information priority of grey system theory, Reference [9] established MGM (1, N) model with multivariable information by taking the nth component in $x^{(1)}$ as the initial condition in grey differential equation. Taking the nth component in $x^{(1)}$ sequence, Reference [10] established MGM (1, N) model with multivariable information optimizing the modified initial value and background factor $q$ (in form of $z^{(1)} = qx^{(1)}(k + 1)/(1 - q)x^{(1)}(k)$ ($q \in [0,1]$)). However, all of above models are equidistant models. Reference [11] established non-equidistant and multivariable MGM (1, N) model with background value fitted by homogeneous exponential function which is less popular than non-homogeneous exponential function. Therefore, the modeling mechanism of this model has inherent defect. Reference [12] established non-equidistant and multivariable MGM (1, N) model. However, its background value was generated by average, so it is necessary to further improve the precision. With the background value fitted by non-homogeneous exponential function, Reference [13] established non-equidistant and multi-variable MGM (1, N) model to improve the model precision. However, its parameters are only treated by least square method, rather than optimized by corresponding function. Reference [14] analyzed the background value construction method in multivariable grey model MGM (1, m), and utilized vector continued fraction to reconstruct background value by rational interpolation, trapezoid formula and extrapolation. Therefore, it can effectively improve the fitting and prediction accuracy of the model which is always multivariable and non-equidistant MGM (1, m). Based on Reference [15], this work constructed the background value of multivariable and non-equidistant grey model MGM (1, n) with the modeling idea of gradual optimization. In addition, multivariable and non-equidistant grey model MGM (1, n) was established by taking the first component in sequence $x^{(1)}$ as initial condition of grey differential equation and design variable after modification, the minimum mean of relative error as target function. Applying in modeling in both equal-distance and non-equal distance, this model expands the application scope of grey model. Moreover, this model has good theoretical value and application value for its high precision and simple use.

**MODIFIED GREY MODEL MGM (1, N) WITH NON-EQUIDISTANCE AND MULTIVARIABLE BASED ON THE BACKGROUND VALUE BY GRADUAL OPTIMIZATION**

**Definition 1**: Let the sequence $X_{i}^{(0)} = [x_{i}^{(0)}(t_{1}), x_{i}^{(0)}(t_{2}), \cdots, x_{i}^{(0)}(t_{n}), \cdots, x_{i}^{(0)}(t_{m})]$. If $\Delta t = t_{j} - t_{j-1} \neq const, i = 1, 2, \cdots, n, j = 2, \cdots, m$ $n$ is the variable number, and $m$ is the sequence number of every variable, $X_{i}^{(0)}$ can be called as non-equidistant sequence.

**Definition 2**: Let sequence $X_{i}^{(1)} = [x_{i}^{(1)}(t_{1}), x_{i}^{(1)}(t_{2}), \cdots, x_{i}^{(1)}(t_{j}), \cdots, x_{i}^{(1)}(t_{m})]$. If $x_{i}^{(1)}(t_{1}) = x_{i}^{(0)}(t_{1})$, and $x_{i}^{(1)}(t_{j}) = x_{i}^{(1)}(t_{j-1}) + x_{i}^{(0)}(t_{j}) \cdot \Delta t_{j}, j = 2, \cdots, m, i = 1, 2, \cdots, n, \Delta t_{j} = t_{j} - t_{j-1}$, $X_{i}^{(1)}$ can be called as (1-AG0) generated by the first-order accumulation of non-equidistant sequence $X_{i}^{(0)}$. Let the original data matrix of multivariable be
\[
\mathbf{X}^{(0)} = \{x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}\}^T = \begin{bmatrix} x_1^{(0)}(t_1) & x_1^{(0)}(t_2) & \cdots & x_1^{(0)}(t_m) \\ x_2^{(0)}(t_1) & x_2^{(0)}(t_2) & \cdots & x_2^{(0)}(t_m) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(0)}(t_1) & x_n^{(0)}(t_2) & \cdots & x_n^{(0)}(t_m) \end{bmatrix} \tag{1}
\]

where the observed value of all variables in \(\mathbf{X}^{(0)}(t_j)(j = 1, 2, \ldots, m)\) can be expressed as \(\mathbf{X}^{(0)}(t_j) = [x_1^{(0)}(t_j), x_2^{(0)}(t_j), \ldots, x_n^{(0)}(t_j)]\) at time \(t_j\). The sequence \([x_i^{(0)}(t_1), x_i^{(0)}(t_2), \ldots, x_i^{(0)}(t_j), \ldots, x_i^{(0)}(t_m)](i = 1, 2, \ldots, n, j = 1, 2, \ldots, m)\) is not an equidistant sequence that is, the distance \(t_j - t_{j-1}\) is not a constant.

In order to establish the model, a new matrix is generated by accumulating the original data again.

\[
\mathbf{X}^{(1)} = \{x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\}^T = \begin{bmatrix} x_1^{(1)}(t_1) & x_1^{(1)}(t_2) & \cdots & x_1^{(1)}(t_m) \\ x_2^{(1)}(t_1) & x_2^{(1)}(t_2) & \cdots & x_2^{(1)}(t_m) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(1)}(t_1) & x_n^{(1)}(t_2) & \cdots & x_n^{(1)}(t_m) \end{bmatrix} \tag{2}
\]

where \(x_i^{(1)}(t_j)(i = 1, 2, \ldots, n)\) meets with Definition 2 that is,

\[
x_i^{(1)}(t_j) = \sum_{j=1}^{k} x_i^{(0)}(t_j)(t_j - t_{j-1}) \quad (k = 2, \ldots, m)
\]

\[
x_i^{(0)}(t_1) \quad (k = 1)
\]

MGM (1, n) model with multivariable and non-equidistance is an \(n\)-meta first-order differential equation set which can be expressed as follows

\[
\begin{align*}
\frac{dx_1^{(1)}}{dt} &= a_{11}x_1^{(1)} + a_{12}x_2^{(1)} + \cdots + a_{1n}x_n^{(1)} + b_1 \\
\frac{dx_2^{(1)}}{dt} &= a_{21}x_1^{(1)} + a_{22}x_2^{(1)} + \cdots + a_{2n}x_n^{(1)} + b_2 \\
&\vdots \\
\frac{dx_n^{(1)}}{dt} &= a_{n1}x_1^{(1)} + a_{n2}x_2^{(1)} + \cdots + a_{nn}x_n^{(1)} + b_n
\end{align*}
\]

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

and

\[
\begin{bmatrix} b_1 \\
b_2 \\
\vdots \\
b_n \end{bmatrix}
\]

Let

\[
\begin{align*}
A &= \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} \\
B &= \begin{bmatrix} b_1 \\
b_2 \\
\vdots \\
b_n \end{bmatrix}
\end{align*}
\]

Formula (4) can be expressed as follows:

\[
\frac{dX^{(1)}(t)}{dt} = AX^{(1)}(t) + B
\]

The continuous time response of Formula (5) is as follows:

\[
X^{(1)}(t) = e^{At}X^{(1)}(t_0) + A^{-1}(e^{At} - I)B
\]

where \(e^{At} = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} t^k\), \(I\) is a unit matrix.
In order to identify \( A \) and \( B \), the following equation can be obtained by integrating Formula (4) at both sides in the interval \([t_{j-1}, t_j]\).

\[
\begin{align*}
x_i^{(0)}(t) & = \sum_{j=1}^{n} a_d \int_{t_{j-1}}^{t_j} x_i^{(1)}(t) \, dt + b_j \Delta t_j (i = 1, 2, \ldots, n; j = 2, 3, \ldots, m) \quad (7) \\
x_i^{(0)}(t) & = \sum_{j=1}^{n} a_d \int_{t_{j-1}}^{t_j} x_i^{(1)}(t) \, dt \quad (8)
\end{align*}
\]

There is always a relatively large error because \( \int_{t_j}^{t_{j+1}} f(t) \, dt \Delta t_j \text{, while the traditional equation of background value is actually for the trapezoid area } \int_{t_j}^{t_{j+1}} f(t) \, dt \text{. Therefore, the parameter matrix } \hat{A} \text{ and parameter } \hat{B} \text{ which are predicted with background value in the interval } [t_{j-1}, t_j] \text{ by taking } \int_{t_j}^{t_{j+1}} x_i^{(1)}(t) \, dt = \frac{\Delta t_j}{t_{j+1} - t_j} \text{. According to the accurate index rule of grey prediction model, all the known data can be determined by grey modelling when } x_i^{(1)}(t) = a_i e^{b_i t} + c_i \text{, where } a_i, b_i, c_i \text{ are all undetermined coefficients meeting with } x_i^{(1)}(t) = b_j e^{a_j (t_j - t_i)} + c_j \text{. Reference } [15] \text{ analyzed the idea and method of GM (1, 1) model, and concluded that the key to establishing a model is the winterization derivative } \frac{dx_i^{(1)}(t)}{dt} \text{. The modeling accuracy can only be improved when the winterization derivative is rationally selected. The most perceptual and understandable way of winterization derivative is replacing the differential number with difference quotient which can be expressed as follows:}
\]

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
</table>
| \[
\frac{dx_i^{(1)}(t)}{dt} \approx \frac{x_i^{(1)}(t_{j+1}) - x_i^{(1)}(t_j)}{t_{j+1} - t_j} \quad (8)
\] | |
| \[
\frac{dx_i^{(1)}(t_{j+1})}{dt} \approx \frac{x_i^{(1)}(t_{j+1}) - x_i^{(1)}(t_j)}{t_{j+1} - t_j} \quad (9)
\] | |

With the precondition that there will be the first-order derivative \( x(t) \), it can be known according to Lagrange’s mean value theorem that \( \frac{x_i^{(1)}(t_{j+1}) - x_i^{(1)}(t_j)}{t_{j+1} - t_j} \) must be the derivative \( x_i'(t_j) \) at one point in \((t_k, t_{k+1})\). In other words, the derivative can be treated as known, and the corresponding independent variable is the grey number in interval \((t_j, t_{j+1})\). Therefore, the correction factor \( \rho \) is introduced to
correct the grey coefficient, and the model is established with $\rho \frac{x_i(t_{j+1}) - x_i(t_j)}{t_{j+1} - t_j}$. However, $\xi_i(t_j)$ and $\rho_i(t_j)$ which are unknown should be obtained by gradual optimization in following steps:

(1) For Data $(2)$, the initial value of iteration number is $j = 0$, and the initial value is $a_j = 0$.

(2) The initial value of accumulated iteration number is $j = 0$, and the initial value is $a_j = 0$. Therefore,

$$
\xi_i(t_j) = \frac{e^{a_j} - 1}{e^{a_j} - 1} - \frac{1}{2}, \quad \rho_i(t_j) = \frac{a(1 + e^{-a})}{2(1 - e^{-a})} = 1
$$

For the winterization value of $(x_i(t_j), x_j^*(t_j))$, the winterization value of parameter $a$ will be obtained by linear regression of $(1 - \xi(t_j), x_i(t_j))$.

\begin{align}
\hat{a}_i(t_{j+1}) &= -\rho_i(t_j) \frac{s_{sys}}{s_{xxs}} 
\end{align}

where

\begin{align}
\bar{X}_j &= \frac{1}{m-1} \sum_{j=1}^{m-1} x_i(t_j), \quad \bar{y} = \frac{1}{m-1} \sum_{j=1}^{m-1} y_j, \quad y_j = \frac{x_i(t_{j+1}) - x_i(t_j)}{t_{j+1} - t_j} \\
x_{ji} &= [1 - \xi(t_j)]x_i(t_j) + \xi(t_j)x_i(t_{j+1}), \quad s_{xxs} = \sum_{j=1}^{m-1} [x_{ji} - \bar{X}_j]^2 \\
s_{sys} &= \sum_{j=1}^{m-1} (x_{ji} - \bar{X}_j)(y_j - \bar{y})
\end{align}

Then, the index model $M_{j+1}$ will be obtained by linear regression of $e^{-a_j(t_j - t_1)}, x_i(t_j)$.

\begin{align}
\hat{x}_i(t_{j+1}) &= \hat{c}_{j+1} e^{-\hat{a}_{j+1}(t_{j+1} - t_j)} + \hat{b}_{j+1}
\end{align}

where

\begin{align}
\hat{c}_{j+1} &= \sum_{j=1}^{m} \{[x_i(t_j)] - \frac{1}{m} \sum_{j=1}^{m} x_i(t_j)] \{e^{-\hat{a}_{j+1}(t_{j+1} - t_1)} - \frac{1}{m} \sum_{k=1}^{m} e^{-\hat{a}_{j+1}(t_{j+1} - t_1)}] \} \\
\sum_{j=1}^{m} [e^{-\hat{a}_{j+1}(t_{j+1} - t_1)} - \frac{1}{m} \sum_{k=1}^{m} e^{-\hat{a}_{j+1}(t_{j+1} - t_1)}]^2
\end{align}
\[ \hat{b}_{j+1} = \frac{1}{m} \sum_{k=1}^{m} x_j^{(1)}(t_i) - \hat{c}_{j+1} \frac{1}{m} \sum_{j=1}^{m} e^{-\hat{a}_{j+1}(t_j-t_i)} \] (13)

\[ z_i^{(1)}(t_j) = \frac{\int_{t_{j-1}}^{t_j} x_i^{(1)}(t_j) \, dt}{\Delta t_j} = x_i^{(1)}(t_j) \] can be obtained after acquiring \( z_i^{(1)}(t_j) \) by \( \hat{a}_j, \hat{b}_j, \hat{c}_j \).

Let \( \mathbf{a}_i = (a_{i1}, a_{i2}, \cdots, a_{in}, b_j)^T (i = 1, 2, \cdots, n) \), the value \( \hat{a}_i \) of \( a_i \) can be obtained by least square method.

\[ \hat{a}_i = [\hat{a}_{i1}, \hat{a}_{i2}, \cdots, \hat{a}_{in}, \hat{b}_i]^T = (\mathbf{L}^T \mathbf{L})^{-1} \mathbf{L}^T \mathbf{y}_i, i = 1, 2, \cdots, n \] (14)

where

\[
\mathbf{L} = \begin{bmatrix}
\pi_1^{(t_2)} & \pi_2^{(t_2)} & \cdots & \pi_n^{(t_2)} & 1 \\
\pi_1^{(t_3)} & \pi_2^{(t_3)} & \cdots & \pi_n^{(t_3)} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\pi_1^{(t_m)} & \pi_2^{(t_m)} & \cdots & \pi_n^{(t_m)} & 1
\end{bmatrix}
\] (15)

\[ \mathbf{y}_i = [x_i^{(0)}(t_2), x_i^{(0)}(t_3), \cdots, x_i^{(0)}(t_m)]^T \] (16)

Therefore, the identification values of \( \mathbf{A} \) and \( \mathbf{B} \) can be obtained.

\[
\hat{\mathbf{A}} = \begin{bmatrix}
\hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\
\hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{a}_{n1} & \hat{a}_{n2} & \cdots & \hat{a}_{nn}
\end{bmatrix}
\]

\[
\hat{\mathbf{B}} = \begin{bmatrix}
\hat{b}_1 \\
\hat{b}_2 \\
\vdots \\
\hat{b}_n
\end{bmatrix}
\] (17)

The equation for MGM (1, n) model is as follows

\[ \hat{X}_i^{(1)}(t_j) = e^{\hat{\mathbf{A}}(t_j-t_i)} \hat{X}_i^{(1)}(t_i) + \hat{\mathbf{A}}^{-1}(e^{\hat{\mathbf{A}}(t_j-t_i)} - 1)\hat{\mathbf{B}}, j = 1, 2, \cdots, m \] (18)

Above equation takes the solution to the data in the first row as initial value. In addition, it makes modification by replacing \( X_i^{(0)}(t_i) \) with \( X_i^{(0)}(t_i) + \beta_i \) where, \( \beta_i \) is the vector equivalent to variable dimension that is, \( \beta = [\beta_1, \beta_2, \cdots, \beta_n]^T \). The fitted value of restored original data is as follows:

\[ \hat{X}_i^{(0)}(t_i) = X_i^{(0)}(t_i) + \beta_i \]

\[ \hat{X}_i^{(0)}(t_j) = (\hat{X}_i^{(1)}(t_j) - \hat{X}_i^{(1)}(t_{j-1}))/ (t_j - t_{j-1}), j = 2, 3, \cdots, m \] (19)
The absolute error of \(i\)th variable is defined as
\[
\varepsilon_i(t_j) = \hat{x}_i^{(0)}(t_j) - x_i^{(0)}(t_j)
\]

The relative error of \(i\)th variable (\%) is
\[
\frac{\varepsilon_i(t_j)}{x_i^{(0)}(t_j)} \times 100
\]

The mean of relative error of \(i\)th variable is
\[
\frac{1}{m} \sum_{j=1}^{m} |\varepsilon_i(t_j) |
\]

The average error of all data is
\[
f = \frac{1}{nm} \sum_{i=1}^{n} \left( \sum_{j=1}^{m} |\varepsilon_i(t_j) | \right)
\]

MODEL APPLICATION EXAMPLES

**Example 1** refers to the data about influences of water absorption on pure PA66 mechanical property. The experimental data, including bending strength, modulus of elasticity in static bending, and tensile strength of PA66 was obtained by conducting mechanical property test to PA66 samples with different water absorption.

<table>
<thead>
<tr>
<th>No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water absorption (t_j)/%</td>
<td>0</td>
<td>0.0607</td>
<td>0.1071</td>
<td>0.1662</td>
<td>0.2069</td>
<td>0.4344</td>
<td>0.5243</td>
<td>0.8524</td>
<td>0.9756</td>
</tr>
<tr>
<td>(X_1^{(0)})</td>
<td>83.4</td>
<td>84.9</td>
<td>84.5</td>
<td>84.2</td>
<td>84.4</td>
<td>78.4</td>
<td>75.4</td>
<td>59.5</td>
<td>54.1</td>
</tr>
<tr>
<td>(X_2^{(0)})</td>
<td>2.63</td>
<td>2.64</td>
<td>2.61</td>
<td>2.65</td>
<td>2.66</td>
<td>2.52</td>
<td>2.32</td>
<td>1.90</td>
<td>1.72</td>
</tr>
<tr>
<td>(X_3^{(0)})</td>
<td>84.2</td>
<td>84.4</td>
<td>86.3</td>
<td>84.3</td>
<td>81.3</td>
<td>74.9</td>
<td>75.7</td>
<td>73.2</td>
<td>66.9</td>
</tr>
</tbody>
</table>

Non-equidistant modified model MGM (1, 3) was established by the method in this work. The parameters of model are as follows:
The fitted value of $X^{(0)}_3$ is as follows:
$$X^{(0)}_3 = [84.2981, 85.2018, 83.1227, 81.6996, 80.3923, 77.1071, 73.5063, 69.418, 65.5712]$$

The absolute error of $X^{(0)}_3$ is as follows:
$$q = [0.09809, 0.80176, -3.1773, -2.6004, -0.90768, 2.2071, -2.1937, -3.782, -1.3288]$$

The relative error (%) of $X^{(0)}_3$ is as follows:
$$e = [0.1165, 0.94995, -3.6817, -3.0847, -1.1165, 2.9467, -2.8978, -5.1666, -1.9863]$$

The mean of relative error is 2.4385%. Therefore, the accuracy of this model is high.

**Example 2:** In the calculation of contact strength, parameters, the factors $m_a$ and $m_b$ that principal curvatures $F(\rho)$ contacting with long and short radius of ellipse at point always refer to the table. The data abstract is shown in Table 1 [16].

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$F(\rho)$</th>
<th>$m_a$</th>
<th>$m_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>0.99</td>
<td>23.9</td>
<td>7.25</td>
</tr>
<tr>
<td>0.95</td>
<td>0.99</td>
<td>18.5</td>
<td>7.02</td>
</tr>
<tr>
<td>0.99</td>
<td>0.90</td>
<td>14.2</td>
<td>6.84</td>
</tr>
<tr>
<td>0.99</td>
<td>0.80</td>
<td>12.2</td>
<td>6.64</td>
</tr>
<tr>
<td>0.99</td>
<td>0.70</td>
<td>11.0</td>
<td>6.47</td>
</tr>
<tr>
<td>0.99</td>
<td>0.60</td>
<td>10.1</td>
<td>6.33</td>
</tr>
<tr>
<td>0.99</td>
<td>0.50</td>
<td>9.46</td>
<td>6.19</td>
</tr>
<tr>
<td>0.99</td>
<td>0.40</td>
<td>8.92</td>
<td>6.06</td>
</tr>
<tr>
<td>0.99</td>
<td>0.30</td>
<td>8.47</td>
<td>5.95</td>
</tr>
<tr>
<td>0.99</td>
<td>0.20</td>
<td>8.10</td>
<td>5.83</td>
</tr>
<tr>
<td>0.99</td>
<td>0.10</td>
<td>7.67</td>
<td>5.72</td>
</tr>
<tr>
<td>0.99</td>
<td>0.00</td>
<td>7.49</td>
<td>5.63</td>
</tr>
</tbody>
</table>

Let the factor $m_b$ of elliptic short radius $b$ be $t_j$, principal curvature function $F(\rho)$ be $x_i$, and the factor $m_a$ of elliptic long radius $a$ be $x^2_i$. The non-equidistant modified model MGM (1, 2) was established by the method in this work. The parameters of this model are as follows:

$$A = \begin{bmatrix} 0.1155 & -4.2174 & 0.0172 \\ 0.0046 & -0.1662 & 0.0007 \\ -0.0291 & 1.0613 & -0.0043 \end{bmatrix}, \quad B = \begin{bmatrix} 185.1780 \\ 6.1876 \\ 97.9652 \end{bmatrix}, \quad \beta = 3.51 \times 10^{-7}$$

The fitted value of principal curvature function $F(\rho)$ is as follows:
$$\hat{F}(\rho) = \begin{bmatrix} 0.99664 \\ 1.0001 \\ 0.99954 \end{bmatrix}$$

The absolute error of principal curvature function is as follows:
$$q = 10^{-3} \times [-2.86, 1.06, 1.54, 1.02, 0.47, -0.01, -0.39, -0.73, -0.98, -1.13, -1.23, -1.27, -1.41, -0.95, -0.88, -0.83, -0.58, -0.34, -0.22, 0.21, 0.41, 0.73]$$

Relative error of principal curvature function (%) is as follows
\[ e = 10^{-3} \times [-0.28,0.11,0.15,0.10,0.05,0,-0.04,-0.07,-0.10,-0.11,-0.13,-0.13,-0.12,-0.10,-0.09,-0.08,-0.06,-0.03,-0.02,0.02,0.04,0.07] \]

The mean of relative error is 0.085031%. Therefore, the accuracy of this model is high.

**CONCLUSIONS**

The background value of non-equidistant and multivariable modified grey model MGM (1, n) was constructed based on the gradual optimization modelling for non-equidistant and multivariable sequence with mutual effect and interaction among variables. A non-equidistant and multivariable modified grey model MGM (1, n) was established, with minimum mean of relative error as target function and the correction of corresponding initial value for non-equidistant and multivariable grey model MGM (1, n) as the design variable. Applying in both equidistant and non-equidistant modelling, this new model expands the application scope of grey model for its high precision and simple use. Moreover, examples show the practicability and reliability of the model. With practical and theoretical significance, this model is worth promoting and utilizing.

**ACKNOWLEDGMENTS**

This research is supported by the grant of the 12th Five-Year Plan for the construct program of the key discipline (Mechanical Design and Theory) in Hunan province(XJF2011[76]), and Cooperative Demonstration Base of Universities in Hunan, “R & D and Industrialization of Rock Drilling Machines” (XJT [2014] 239).

**REFERENCES**


